Fourier analysis of equivariant quantum cohomology II

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§ Quantum $\mathcal{D}$-module

X : smooth projective variety with $T_{\mathbb{C}}$-action $(T_{\mathbb{C}} \cong (\mathbb{C}^\times)^\ell)$

- $\mathbb{C}[\![Q]\!] = \left\{ \sum_{\underline{d \in NE_{\mathbb{N}}(X)}} a_d Q^d \;\middle|\; a_d \in \mathbb{C} \right\}$

  effective classes in $H_2(X,\mathbb{Z})$

  Novikov ring

- $*$: supercommutative, associative multiplication

- $QH_T^*(X) = \left(H_T^*(X)[\![Q,\tau]\!] , * \right)$

  $\tau \in H_T^*(X)$ bulk deformation parameter

$$\underline{(\alpha * \beta, \gamma)} = \sum_{\substack{d \in NE_{\mathbb{N}}(X) \\ n \geq 0}} \langle \alpha, \beta, \gamma, \overbrace{\tau, \dots, \tau}^{n} \rangle^{X,T}_{0,n+3,d} \frac{Q^d}{n!}$$

equivariant Poincaré pairing

$\alpha, \beta, \gamma \in H_T^*(X)$

- $\boxed{QDM_T(X) = H_T^*(X)[z][\![Q,\tau]\!]}$

  quantum D-module

equipped with flat connection

$$\begin{cases} \nabla_{\xi Q \frac{\partial}{\partial Q}} = \xi Q \frac{\partial}{\partial Q} + \frac{1}{z}(\xi *) \\ \nabla_{\tau^\alpha} = \frac{\partial}{\partial \tau^\alpha} + \frac{1}{z}(\phi_\alpha *) \end{cases}$$

- $\bar{\xi} \in H^2(X)$ is the image of $\xi \in H_T^2(X)$
- $\bar{\xi} Q \frac{\partial}{\partial Q}$: derivation of $\mathbb{C}[\![Q]\!]$

$$\left(\bar{\xi} Q \frac{\partial}{\partial Q}\right) Q^d = (\bar{\xi} \cdot d)\, Q^d$$

- $E = c_1^T(X) + \sum_\alpha \left(1 - \frac{1}{2}\deg \phi_\alpha\right) \tau^\alpha \phi_\alpha$

  Euler vector field

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(E*) + \mu$$

$\mu(\phi_\alpha) = \left(\frac{1}{2}\deg\phi_\alpha - \frac{n}{2}\right)\phi_\alpha$: not $\mathbb{C}[X]$ - linear

where $\xi \in H^2_T(X)$, $\tau = \sum_\alpha \tau^\alpha \phi_\alpha$ $\{\phi_\alpha\}$: $\mathbb{C}$-basis of $H^*_T(X)$

Rem ∞ many parameters $\{\tau^\alpha\}$

Rem can work with "graded completion"

$\deg \tau^\alpha = 2 - \deg\phi_\alpha$ $\deg z = 2$

$\deg Q^d = 2\, c_1(X)\cdot d$

- Pairing $f, g \in QDM_T(X)$

$$P(f, g) = (f(-z), g(z))$$: compatible with ∇

§ Shift operator

Recall $k \in \mathrm{Hom}(S^1, T)$ induces the map

$$\mathcal{L}X \to \mathcal{L}X$$
$$\gamma(e^{i\theta}) \mapsto k(e^{i\theta})\cdot \gamma(e^{i\theta})$$

- But this does NOT canonically lift to $\widetilde{\mathcal{L}X}$

$\exists$

$$0 \to H_2(X,\mathbb{Z}) \to H_2^T(X,\mathbb{Z}) \to H_2^T(pt;\mathbb{Z}) \to 0 \quad \text{exact sequence}$$

$H_2^T(pt;\mathbb{Z}) = \mathrm{Hom}(S^1, T)$

$H_2(X,\mathbb{Z}) \circlearrowright \widetilde{\mathcal{L}X}$: deck transformation, Novikov var

$H_2^T(X,\mathbb{Z}) \circlearrowright \widetilde{\mathcal{L}X}$: $\exists$ combined action

$H_2^T(pt;\mathbb{Z}) \circlearrowright \mathcal{L}X$: Seidel operator S^k

Rem $N_1(X) \subset H_2(X;\mathbb{Z})$ algebraic classes

$$0 \to N_1(X) \to N_1^T(X) \to H_2^T(pt;\mathbb{Z}) \to 0$$

Q^d

Actual Definition "count hol sections of $E_k \to \mathbb{P}^1$ (Seidel fibration)"

$$E_k = X \times (\mathbb{C}^2 \setminus 0) \big/ (x, (v_1, v_2)) \sim (s^k x, (s v_1, s v_2)) \quad : X\text{-fibration over } \mathbb{P}^1$$

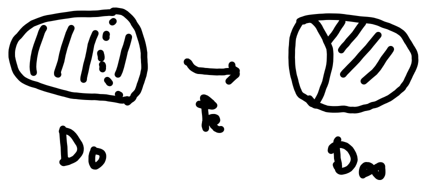
$$s \in \mathbb{C}^\times \qquad s^k := k(s)$$

$$\begin{tikzcd} X \arrow[r, hook] & E_k \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

Rem $E_k = (X \times D_0^2) \cup_\varphi (X \times D_\infty^2)$

$\varphi : X \times \partial D_0^2 \to X \times \partial D_\infty^2$, $(x, e^{i\theta}) \mapsto (k(e^{-i\theta})x, e^{-i\theta})$ clutching function

Sections of E_k $\longleftrightarrow$ two discs in X whose bdries are related by k



We have a fiber diag

$$\begin{tikzcd} E_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_T = X \times_T ET \arrow[d, "\text{Borel construction}"] \\ \mathbb{P}^1 \subset \mathbb{P}^\infty \arrow[r, "Bk"] & BT \end{tikzcd}$$

$\rightsquigarrow$ section of $E_k \to \mathbb{P}^1$ defines an equivariant class in $H_2^T(X; \mathbb{Z})$ (actually in $N_1^T(X)$) lying over $k \in H_2^T(pt; \mathbb{Z})$

BS

- $\hat{T}_{\mathbb{C}} := T_{\mathbb{C}} \times \mathbb{C}^*$ action

$$(\lambda, z) \cdot [x, (v_1, v_2)] = [\lambda \cdot x, (v_1, z v_2)]$$

$$\begin{cases} (\lambda, z) \cdot x = \lambda \cdot x & \text{for } x \in X_0 \ : \text{ fiber at } 0 \\ (\lambda, z) \cdot x = \lambda z^{k} \cdot x & \text{for } x \in X_\infty \ : \text{ fiber at } \infty \end{cases}$$

different $\hat{T}_{\mathbb{C}}$-action!

- Canonical identification

$\exists\ \Phi_k : H^*_{\hat{T}}(X_0) \cong H^*_{\hat{T}}(X_\infty)$

s.t.

$$\Phi_k(f(\lambda, z)\alpha) = f(\lambda - kz)\,\Phi_k(\alpha)$$

$f(\lambda, z) \in \mathbb{C}[\lambda, z]$

- For $\beta \in N_1^T(X) < H_2^T(X, \mathbb{Z})$, set $k := -\bar{\beta} \in H_2^T(pt; \mathbb{Z})$

$\hat{\mathbb{S}}^\beta : H^*_{\hat{T}}(X_0) \longrightarrow H^*_{\hat{T}}(X_0)$ is defined by

$$\boxed{\left(\Phi_k \hat{\mathbb{S}}^\beta(c_0),\ c_\infty\right)_{H^*_{\hat{T}}(X_\infty)} = \sum_{\substack{d \in N_1(X) \\ n \geq 0}} \left\langle 1_{0*} c_0,\ 1_{\infty *} c_\infty,\ \hat{\tau}, \dots, \hat{\tau} \right\rangle^{E_k, \hat{T}}_{0, n+2, d-\beta} \frac{Q^d}{n!}}$$

$$\begin{cases} c_0 \in H^*_{\hat{T}}(X_0) \\ c_\infty \in H^*_{\hat{T}}(X_\infty) \end{cases}$$

$\hat{\tau} \in H^*_{\hat{T}}(E_k)$ is such that

$$\hat{\tau}|_{X_0} = \tau, \quad \hat{\tau}|_{X_\infty} = \Phi_k(\tau)$$

$d - \beta \in H_2^T(X; \mathbb{Z})$

lies over k

(my section class of E_k)

Claim

- $\hat{\mathbb{S}}^\beta \circ \lambda_i = (\lambda_i - (\lambda_i \cdot \bar{\beta}) z) \circ \hat{\mathbb{S}}^\beta$ ☺ because of Ξ_k

- $\hat{\mathbb{S}}^\beta : QDM_T(X) \longrightarrow Q^{\beta + \sigma_0(-\bar{\beta})} QDM_T(X)$ piecewise linear splitting of $H_2^T(X) \longrightarrow H_2^T(pt)$, $\sigma_0(\cdot)$

  (may involve negative powers of Q)

- $\hat{\mathbb{S}}^\beta = Q^d$ if $\beta = d \in N_1(X)$

- $\hat{\mathbb{S}}^{\beta_1} \circ \hat{\mathbb{S}}^{\beta_2} = \hat{\mathbb{S}}^{\beta_1 + \beta_2}$

§ Fundamental Solution

$M(\tau) \in \mathrm{End}(H_T^*(X))[\![z^{-1}]\!][\![Q, \tau]\!]$ defined by $(M(\tau)\phi_1, \phi_2)$

$$= (\phi_1, \phi_2) + \sum_{\substack{d \in NE_N(X) \\ n \geq 0}} \left\langle \phi_1, \tau, \ldots, \tau, \frac{\phi_2}{z - \psi} \right\rangle_{0, n+2, d} \frac{Q^d}{n!}$$

$$\frac{\phi_2}{z-\psi} = \sum_{b=0}^{\infty} \phi_2 \frac{\psi^b}{z^{b+1}}$$

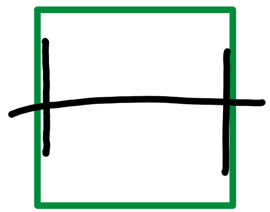
(**Rem** Coeff of $Q^d \tau^{\alpha_1} \cdots \tau^{\alpha_m}$ is a rational fcn of (λ, z))

Prop (Dubrovin, Dijkgraaf, Givental ; Okounkov-Pandharipande, ...)

- $M(\tau)\circ\nabla_{\xi Q\partial_Q} = \left(\xi Q\partial_Q + z^{-1}\xi v\right)\circ M(\tau)$
- $M(\tau)\circ\nabla_{\tau^\alpha} = \partial_{\tau^\alpha}\circ M(\tau)$
- $M(\tau)\circ\nabla_{z\partial_z} = \left(z\partial_z - z^{-1}c_1^T(X)+\mu\right)\circ M(\tau)$
- $M(\tau)\circ\hat{\mathbb{S}}^\beta = \underbrace{\hat{\mathbb{S}}^\beta}\circ M(\tau)$

"hypergeometric" operator
given by classical topology

localization formula



$\hat{T}$- fixed section

$$\hat{\mathbb{S}}^\beta : H^*_{\hat T}(X)_{loc} \circlearrowleft$$

$$\parallel$$

$$H^*_T(X)\otimes_{\mathbb{C}[\lambda]}\mathbb{C}(\lambda,z)$$

$$\hat{\mathbb{S}}^\beta f\Big|_F = Q^{\beta+\sigma_F(-\bar\beta)}\left(\prod_\alpha\prod_j\frac{\prod_{c\le 0}\rho_{F,\alpha,j}+\alpha+cz}{\prod_{c\le -\alpha\cdot\bar\beta}\rho_{F,\alpha,j}+\alpha+cz}\right)$$

$$\times\left(e^{-z\bar\beta\partial_\lambda} f\big|_F\right)$$

for T- fixed component F

shift of equiv parameters

$\lambda_i \longrightarrow \lambda_i - z(\bar\beta\cdot\lambda_i)$

- $N_F = \bigoplus_\alpha N_F^\alpha$

  T-wt decomposition

- $\{\rho_{F,\alpha,j}\}_j$: Chern roots of N_F^α

(Rem $\hat{\mathbb{S}}^\beta$ can be further trivialized by

(the equivariant $\hat{\Gamma}$-class $\Gamma(1+x) = x\Gamma(x)$)

Def (Givental cone)

$$\mathcal{L}_X^{eq} := \bigcup_{\tau \in H_T^*(X)} z M(\tau)\big(H_T^*(X)[z][\![Q]\!]\big)$$

poly in z

⟨ $\hat{\mathbb{S}}^\beta$ preserves this Givental cone ⟩

- Lagrangian subsurfa
- graph (dF_0)
- geometric incarnation of $QDM_T(X)$

§ Equivariant Ample Cone

(Assume: generic stabilizer of $T_{\mathbb{C}} \curvearrowright X$ is finite)

By definition of $\hat{\mathbb{S}}^\beta$, $\exists$ monoid $NE_N^T(X) := NE_N(X) + \langle -\sigma_0(-k) \mid k \in H_2^T(pt; \mathbb{Z}) \rangle_{\mathbb{N}}$

"Equivariant Mori Cone"

s.t. if $\beta \in NE_N^T(X) \Rightarrow \hat{\mathbb{S}}^\beta$ preserves $QDM_T(X)$

(joint work with Fumihiko Sanda)

Prop $QDM_T(X)$ is a module over $\underbrace{\mathbb{C}[z][\![NE_N^T(X)]\!]}_{\text{graded completion}}$ via shift operators

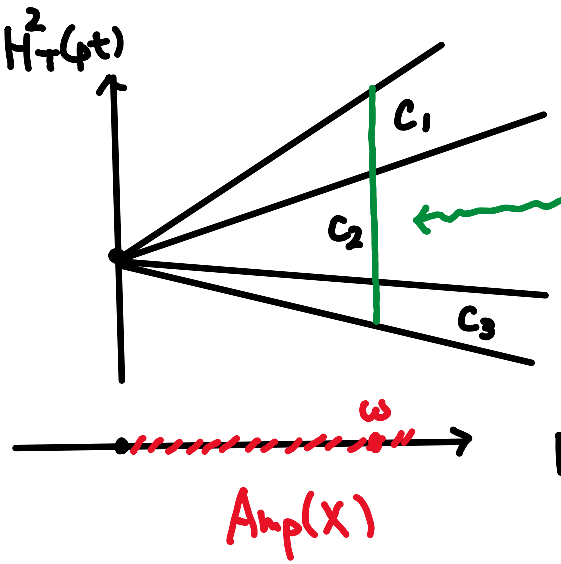
Lemma

$NE^T(X) := \mathbb{R}_{\geq 0} NE_N^T(X) \subset H_2^T(X; \mathbb{R})$

The dual cone of $NE^T(X)$ is the closure of the T-ample cone $C_T(X) \subset H_T^2(X; \mathbb{R})$

(Dolgachev - Hu, Thaddeus)

- $C_T(X) :=$ cone spanned by $c_1^T(L)$ with T-equiv ample line bdl L s.t. $X_{st}(L) \neq \phi$

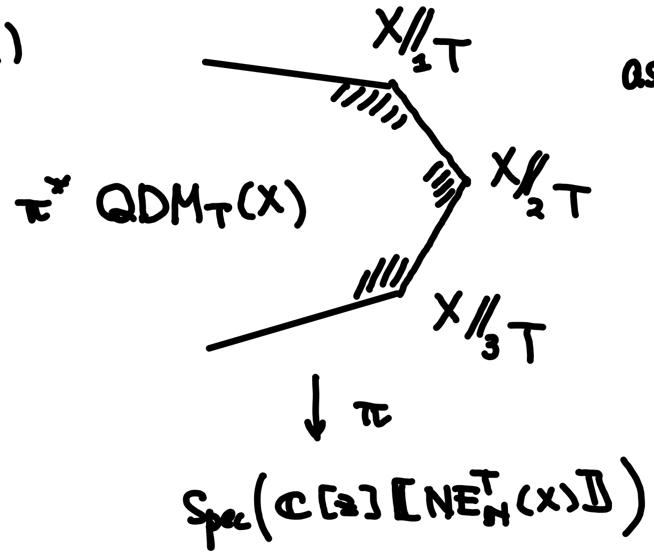


$\pi^{-1}(\omega) \cap C_T(X)$ is the moment polytope of (X, ω)

- $C_T(X)$ is divided into chambers

$$C_T(X) = \bigcup_i C_i$$

- We can regard $QDM_T(X)$ as a sheaf over the "toric variety" associated with this "fan"



extension across each cusp
+
completion
$\rightsquigarrow QDM(X/\!/_i T)$?

<u>Reduction Conjecture</u> (with Sanda)

$X = X/\!/T$ smooth GIT quotient

"Sum over Residues"

Consider the discrete Fourier transform

$J_X^{eq}(\tau) = M(\tau)\,1$ equivariant J-function

(or any element on $\mathcal{L}_X^{eq}$)

$$I := \sum_{[\beta] \in N_1^T(X)/N_1(X)} \kappa\left(\hat{\mathcal{S}}^{-\beta} J_X^{eq}(\tau)\right) \hat{S}^{\beta}$$

where $\kappa : H_T^*(X) \to H^*(Y)$ Kirwan map

Then

① I is supported on the cone $C_Y^\vee$ as an $\hat{S}$ power series ($C_Y \subset C_T(X)$ GIT chamber of Y)

② I lies in the Givental cone of Y

defined over $\mathbb{C}[\![C_{Y,N}^\vee]\!] \xleftarrow[\kappa^*]{} \mathbb{C}[\![Q_Y]\!]$

Novikov ring of Y

The formula for I-fcn : analogous to $f(\lambda) \mapsto \sum_{n \in \mathbb{Z}} f(nz)\, S^n$

<u>Ex 1</u> $X = \mathbb{C}^n$, $T = S^1 \curvearrowright X$ diagonal action

$J^{eq}(0) = 1$

$$\mathcal{S}^k = \frac{\prod_{c \le 0} (\lambda + cz)^n}{\prod_{c \le -k} (\lambda + cz)^n} e^{-kz\partial_\lambda}$$

$$I = \sum_{k \in \mathbb{Z}} \kappa\left(\mathcal{S}^{-k} \cdot 1\right) S^k$$

$$= \sum_{k=0}^{\infty} \frac{S^k}{\prod_{c=1}^{k} (p + cz)^n}$$

($k<0$ terms vanish)

J-function of $\mathbb{P}^{n-1}$

$p = \kappa(\lambda)$

Ex 2 X : toric variety $\curvearrowleft T = (S^1)^n$ $\quad n = \dim_{\mathbb{C}} X$

$$\sum_{k \in \mathbb{Z}^n} \kappa\left(\mathcal{S}^{-k} J_X^{eq}(\tau) \right) S^k = \exp\left({}^{\exists} W(S; Q, \tau, z) / z \right)$$

LG potential mirror to X

(S is the mirror variable)

$\kappa : H_T^*(X) \to H^*(pt) = \mathbb{C}$